Foundations of Discrete Mathematics

Chapters 11 and 12

By Dr. Dalia M. Gil, Ph.D.

Trees

- Tree are useful in computer science, where they are employed in a wide range of algorithms.
- They are used to construct efficient algorithms for locating items in a list.

Trees

- Trees can be used to construct efficient code saving cost in data transmission and storage.
- Trees can be used to study games such as checkers and chess an can help determine winning strategies for playing these games.

Trees

- Trees can be used to model procedures carried out using a sequence of decisions.
- Constructing these models can help determine the computational complexity of algorithms based on a sequence of decisions, such a sorting algorithms.

Trees

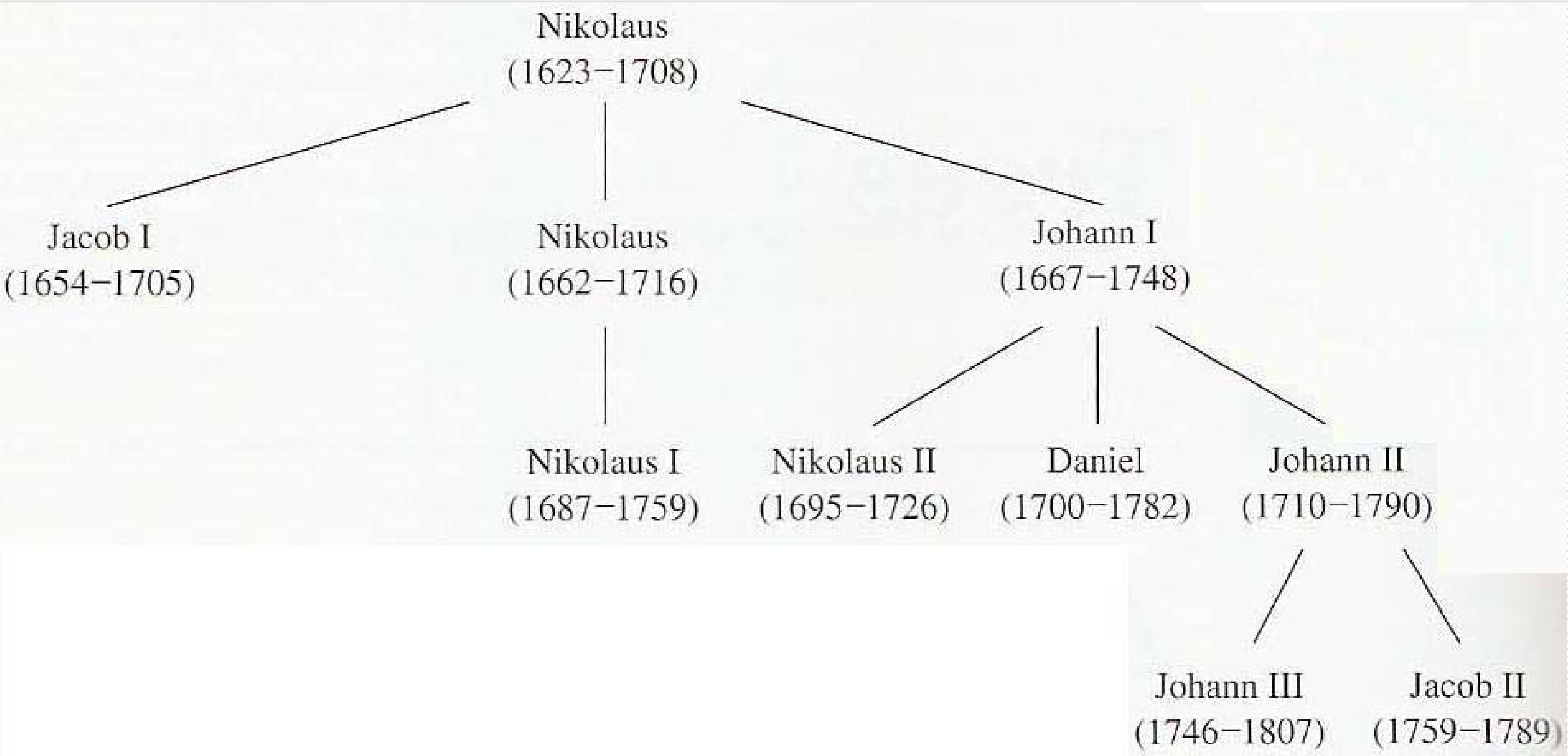
- Procedures for building trees including
- Depth-first search,
- Breadth-first search,

can be used to systematically explore the vertices of a graph.

Trees

- A tree is a connected undirected graph with no simple circuits.
- A tree cannot contain multiple edges or loops.
- Any tree must be a simple graph.

An Example of a Tree



The Bernoulli Family of Mathematicians

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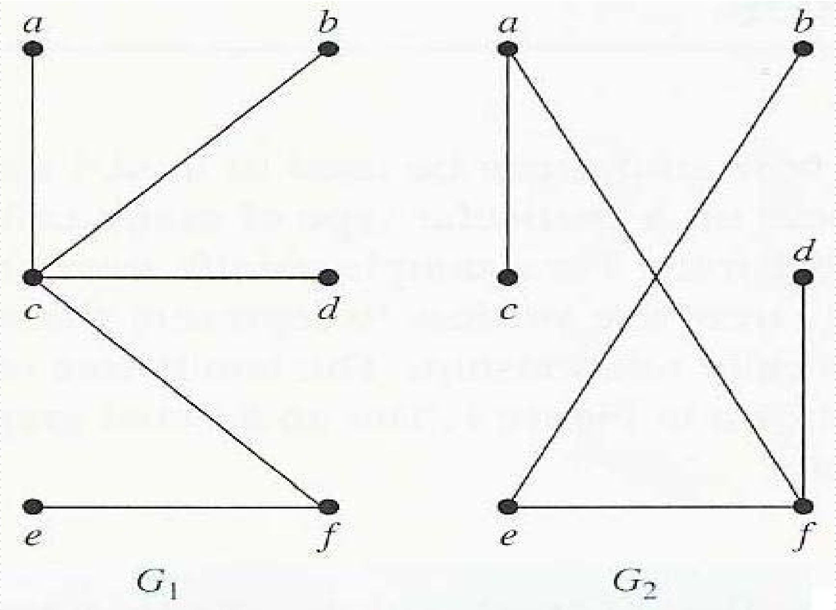
Example: Trees



- G_1 and G_2 are trees.
- Both are connected graph with no simple circuits.

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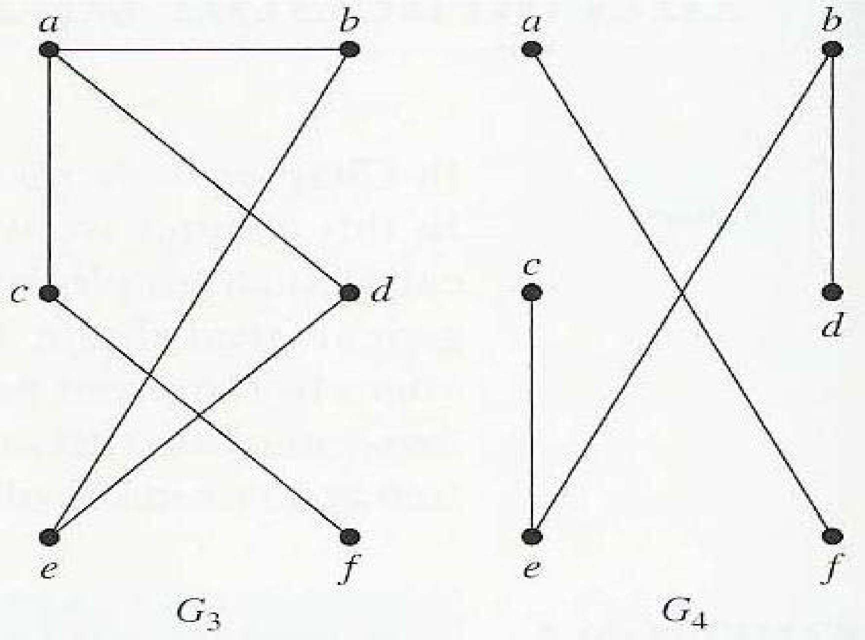
Example: Not Trees



- G_3 is not a tree.

  e, b, a, d, e is a simple circuit.

- G_4 is not a tree. It is not connected.

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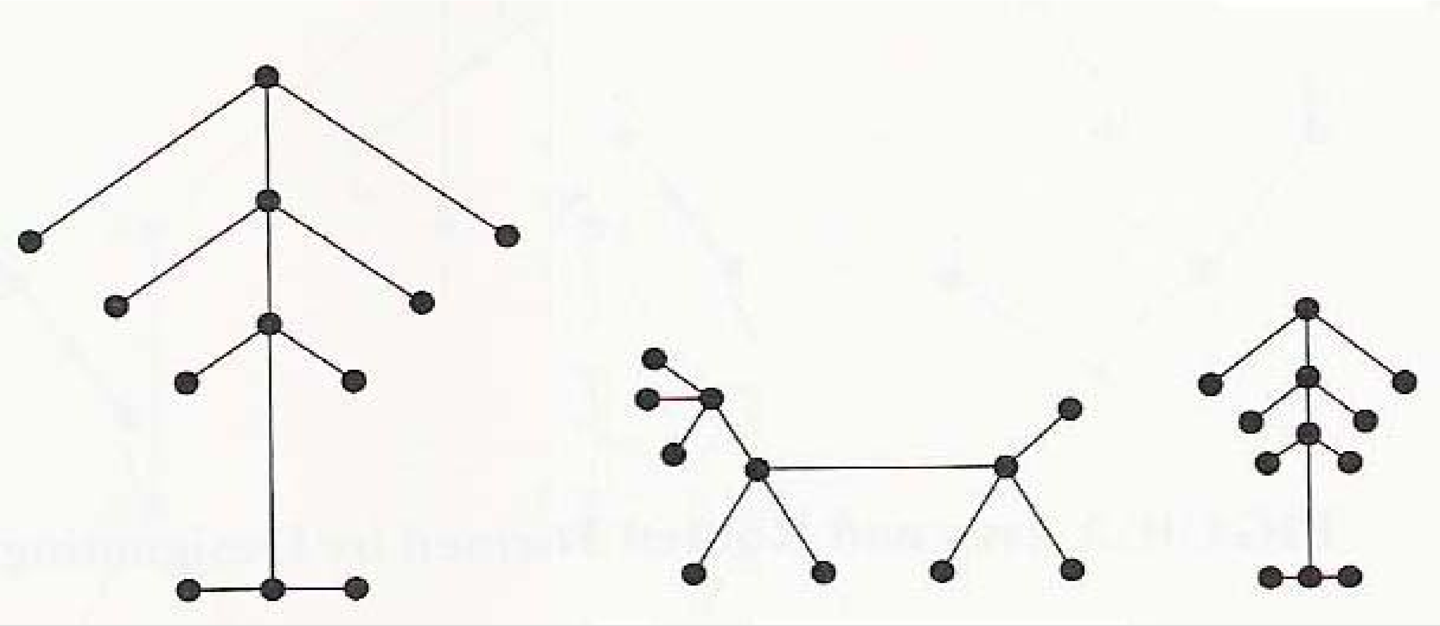
Forest

- A Forest is a graph containing no simple circuits that are not necessarily connected.
- Forests have the property that each of their connected components is a tree.

Example: Forest



A one graph with three connected components

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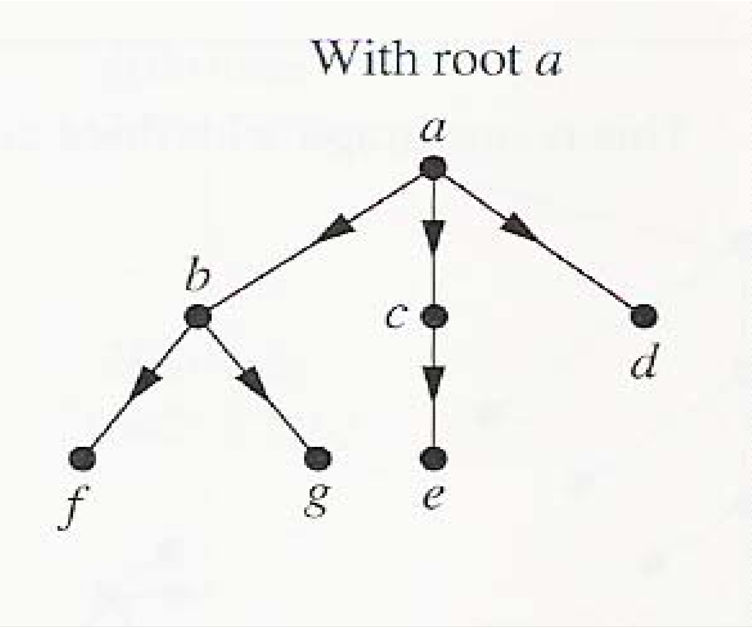
Theorem

- An undirected graph is a tree if and only if there is a unique simple path between any two vertices.

A Rooted Tree



- A rooted tree is a tree in which one vertex has been designated as the root and every edge is directed away from the root.

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Rooted Trees

- We can change an unrooted tree into a rooted tree by choosing any vertex as the root.
- Different choices of the root produce different trees.

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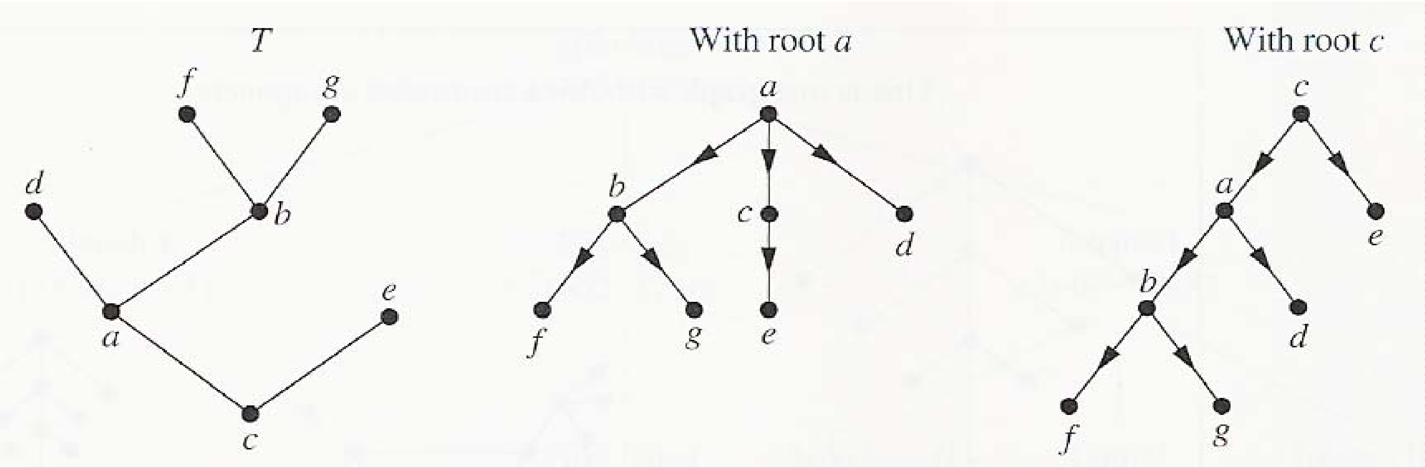
Example: Rooted Trees



- The rooted trees formed by designating a to be the root and c to be the root, respectively, in the tree T.

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The Terminology for Trees

- Suppose that T is a rooted tree. If v is a vertex in T other than the root.
- The **parent** of v is the unique vertex u such that there is a directed edge from u to v.

The Terminology for Trees

- When u is the parent of v, v is called a **child** of u.
- Vertices with the same parent are called **siblings**.

The Terminology for Trees

- The **ancestors** of a vertex other than the root are the vertices in the path from the root to this vertex, excluding the vertex itself and including the root.
- The **descendants** of a vertex v are those that have v as an ancestor.

The Terminology for Trees

- A vertex of a tree is called a **leaf** if it has no children.
- Vertices that have children are called **internal vertices**.
- The root is an internal vertex unless it is the only vertex in the graph, in which case it is a leaf.

The Terminology for Trees

- If ***a*** is a vertex in a tree, the **subtree** with ***a*** as its root is the subgraph of the tree consisting of ***a*** and its descendants and all edges incident to these descendant.

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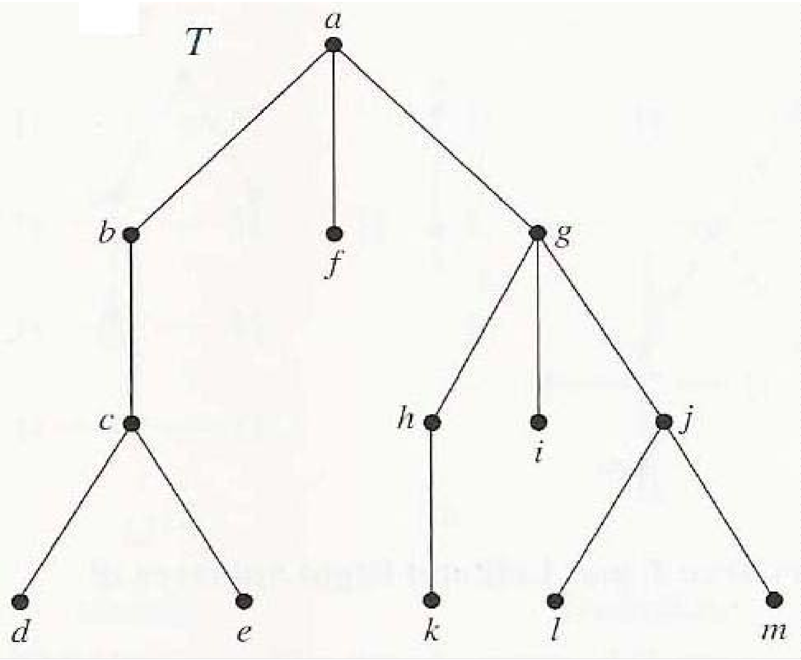
Example: Using Terminology



- T is a rooted tree with root a.
- The parent of vertex c is b.
- The children of g are h, i, and j.

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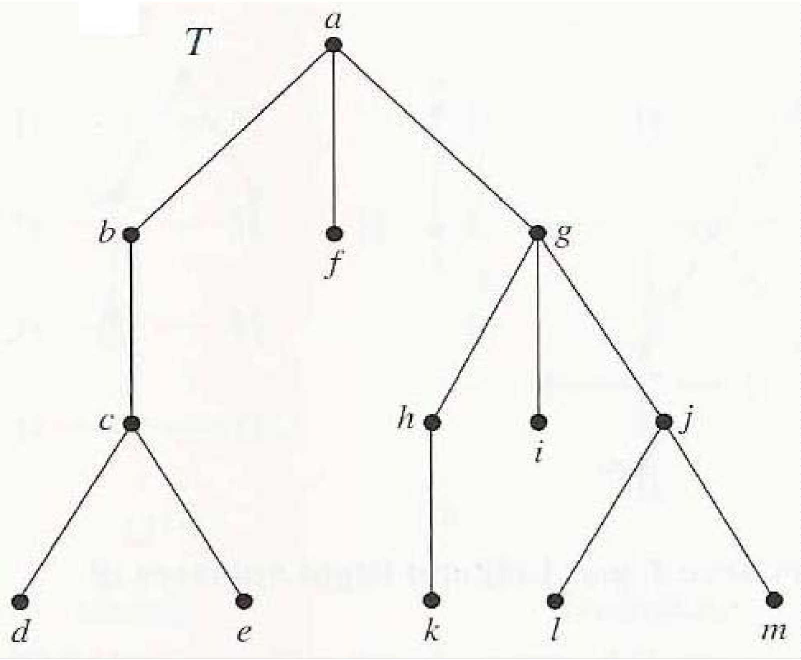
Example: Using Terminology



- The siblings of h are i and j.
- The ancestors of e are c, b, and a.
- The descendant of b are c, d, and e.

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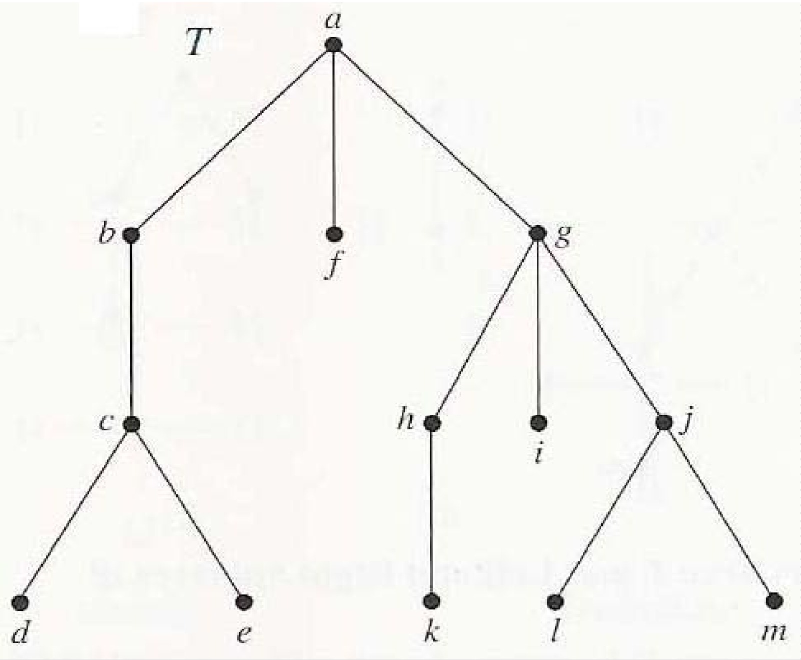
Example: Using Terminology



- ☐ The internal vertices are a, b, c, g, h, and j.
- ☐ The leaves are d, e, f, i, k, l, and m.

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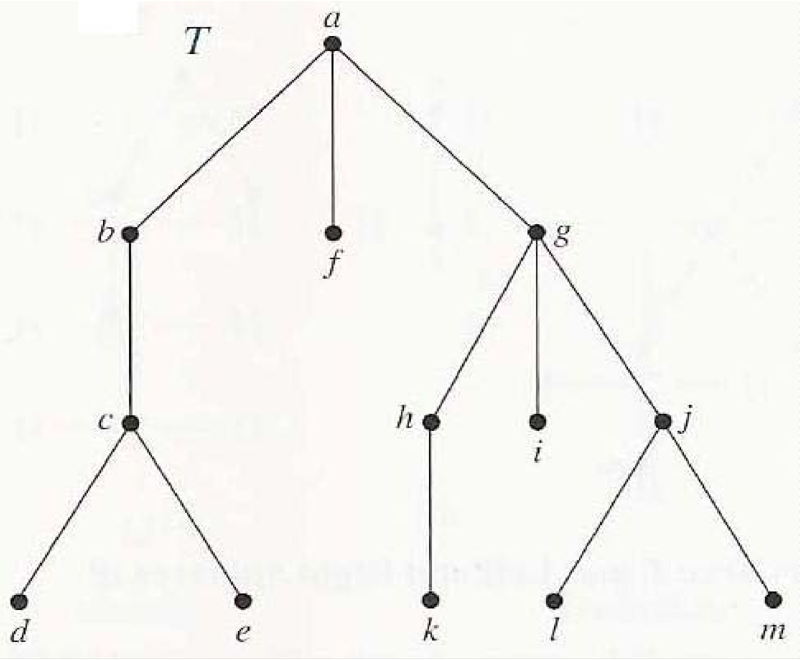
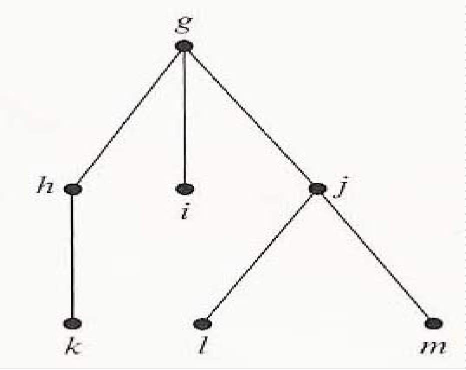
Example: Using Terminology



- The subtree rooted at g is



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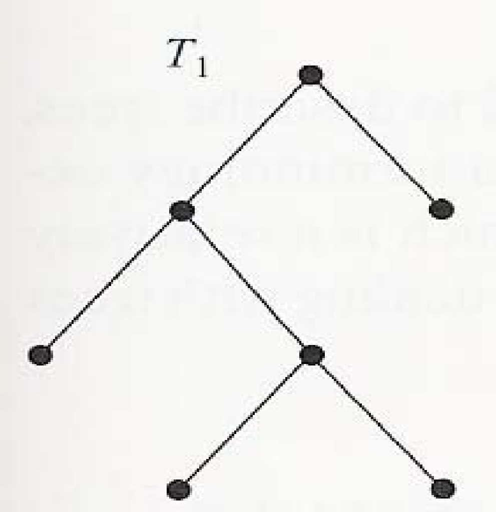
m-ary Tree

- A rooted tree is called m-ary tree if every internal vertex has no more than m children.

- The tree is called a full m-ary tree if every internal vertex has exactly m children.

- An m-ary tree with m=2 is called a binary tree.

Example of m-ary tree



- T_1 is a full binary tree.
- Each of its internal vertices has two children

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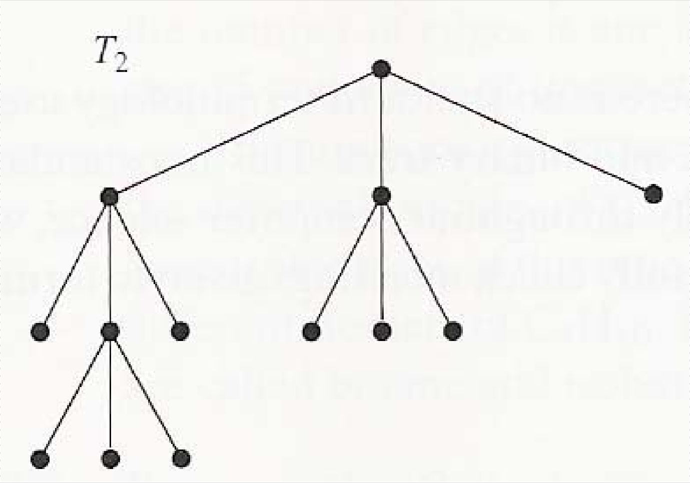
Example of m-ary tree



- T_2 is a full 3-ary tree.
- Each of its internal vertices has three children

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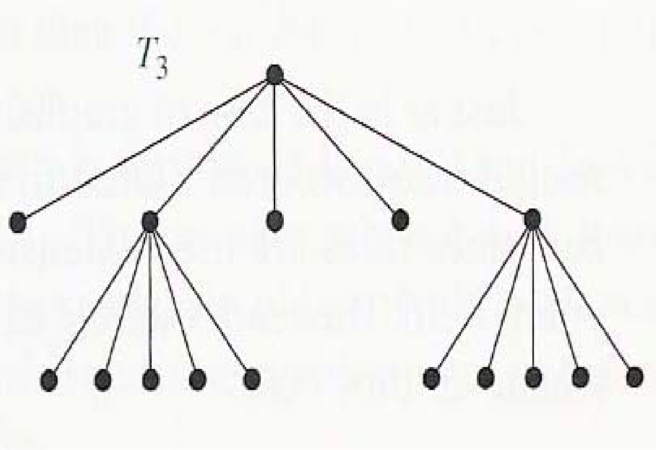
Example of m-ary tree



- T_3 is a full 5-ary tree.
- Each of its internal vertices has five children

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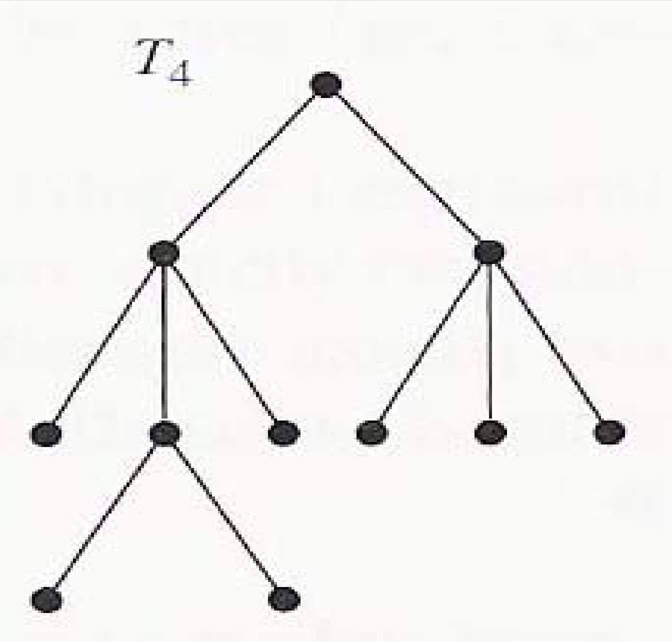
Example of m-ary tree



- T_4 is not a full m-ary tree for any m.
- Some of its internal vertices has 2 children and others have 3.

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Ordered Rooted Tree

- In an ordered rooted tree the children of each internal vertex are ordered.

- Ordered rooted trees are drawn so that the children of each internal vertex are shown in order from left to right.

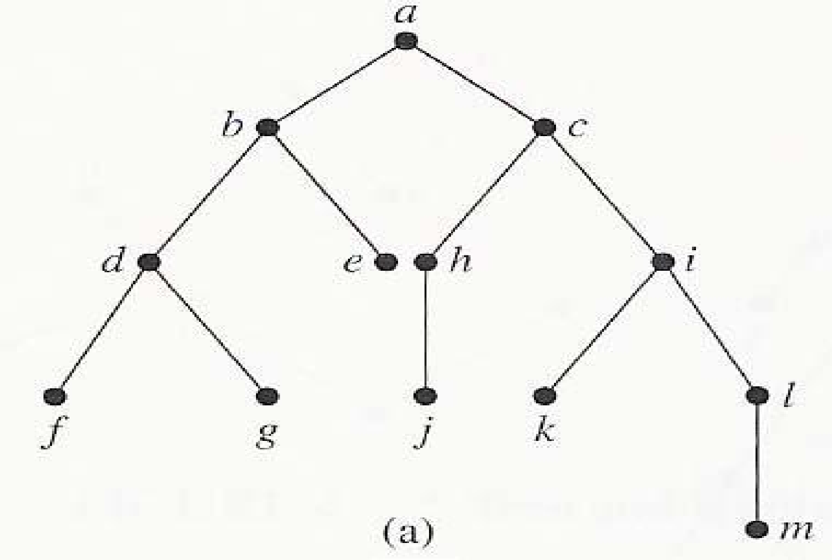
Ordered Binary Tree

- In ordered binary tree (a **binary tree**), an internal vertex has two children.
- The first child is called the **left child** and
- the second child is called the **right child**.

Ordered Binary Tree

- The tree rooted at the left child of a vertex is called the left subtree of this vertex,
- and the tree rooted at the right child of a vertex is called the right subtree of the vertex.

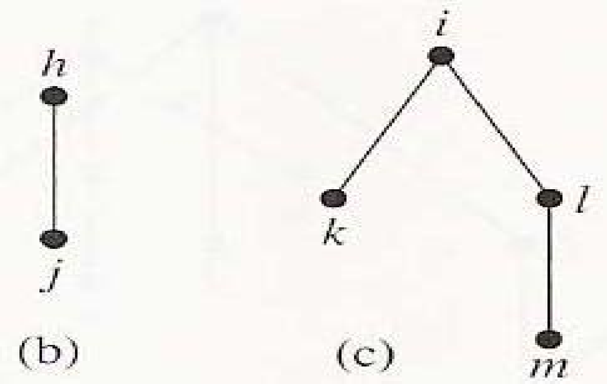
Example



(a)

- The left child of d is f and the right child is g.

- The left and right subtrees of c are



(b) (c)

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Properties of Trees

- A tree with n vertices has n – 1 edges.
- A full m-ary tree with i internal vertices contains $n = m*i + 1$ vertices.
- There are at most m^h leaves in an m-ary tree of hight h.

Where n: vertices, i: internal vertices.

Properties of a Full m-ary Tree

1. n vertices has $i=(n - 1)/m$ internal vertices and $l = [(m - 1) n + 1]/m$ leaves.

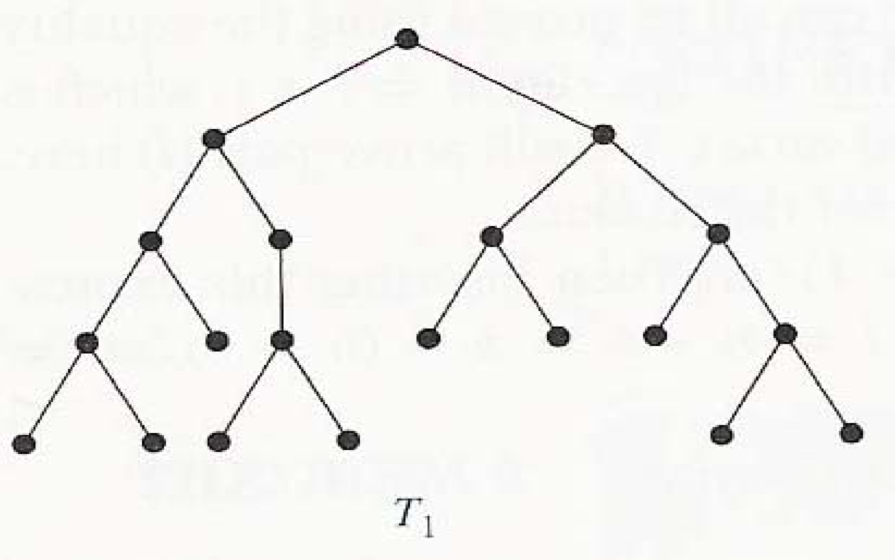
2. i internal vertices has $n = m*i + 1$ vertices and $l = (m - 1)i + 1$ leaves.

3. l leaves has $n = (m*l - 1)/(m - 1)$ vertices and $i = (l - 1)/(m - 1)$ internal vertices.

- l : leaves, m: children, n: vertices, i:int.vertices

Properties of Trees

- A rooted m-ary tree of height ***h*** is balanced if all leaves are at levels h or h – 1.



- T_1 is **balanced.**
- All its leaves are at levels 3 and 4.

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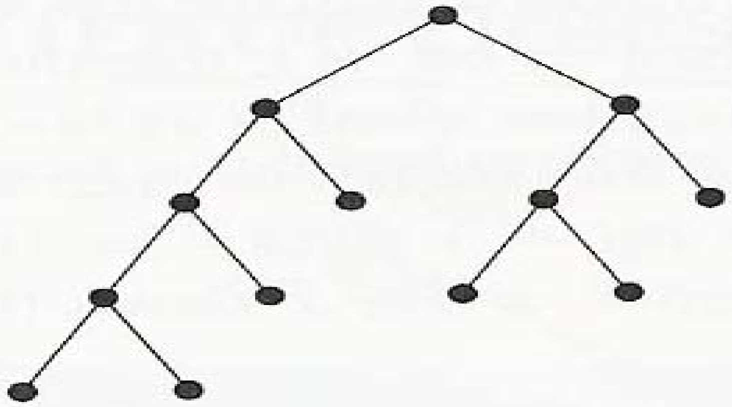
Properties of Trees



T_2

- T_2 is not **balanced**.
- It has leaves at levels 2, 3, and 4.

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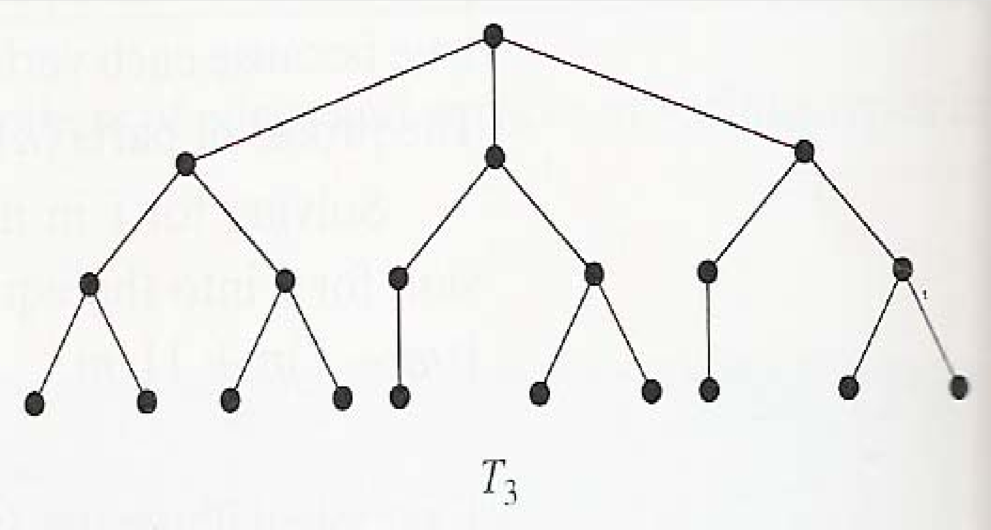
Properties of Trees



T_3

- T_3 is **balanced**.

  All its leaves are at level 3.

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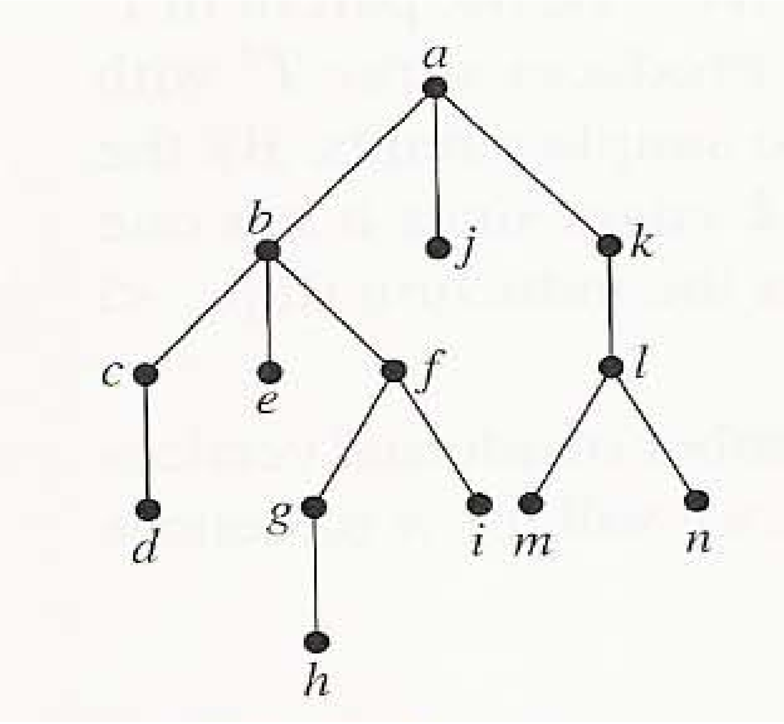
Example: Properties of Trees



- The root a is at level 0.
- Vertices b, j, and k are at level 1.
- Vertices c, e, f, and l are at level 2.

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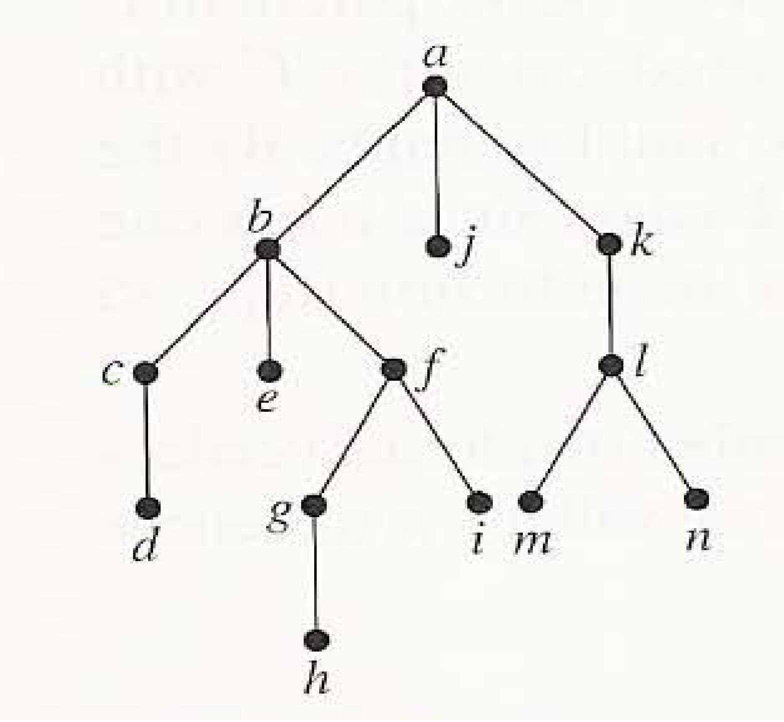
Example: Properties of Trees



- Vertices d, g, i, m, and n are at level 3.
- Vertex h is at level 4.
- This tree has height 4.

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Spanning Trees

- A spanning tree of a connected graph G is a subgraph that is a tree and that includes every vertex of G.

- A minimum spanning tree in a connected weighted graph is a spanning tree that has the smallest possible sum of weights of its edges.

Prim’s Algorithm

- Prim’s algorithm constructs a minimum spanning tree.
- Successively add to the tree edges of minimum weight that are incident to a vertex already in the tree and not forming a simple circuit with those edges already in the tree.
- Stop when n – 1 edges have been adding.

Prim’s Algorithm

- Step 1: Choose any vertex v and let e_1 be and edge of least weight incident with v. Set k = 1.

- Step 2: While $k < n$

  If there exists a vertex that is not in the subgraph T whose edges are $e_1, e_2, \ldots, e_k$,

- Let e_{k+1} be an edge of least weight among all edges of the form ux, where u is a vertex of T and x is a vertex not in T;

Prim’s Algorithm (cont.)

- Let e_{k+1} be an edge of least weight among all edges of the form ux, where u is a vertex of T and x is a vertex not in T;
- Replace k by k + 1;

else output e_1, e_2, …, e_k and stop.

end while.

Example Using Prim’s Algorithm

- Use Prim’s algorithm to design a minimum-cost communication network connecting all the computers represented by the following graph

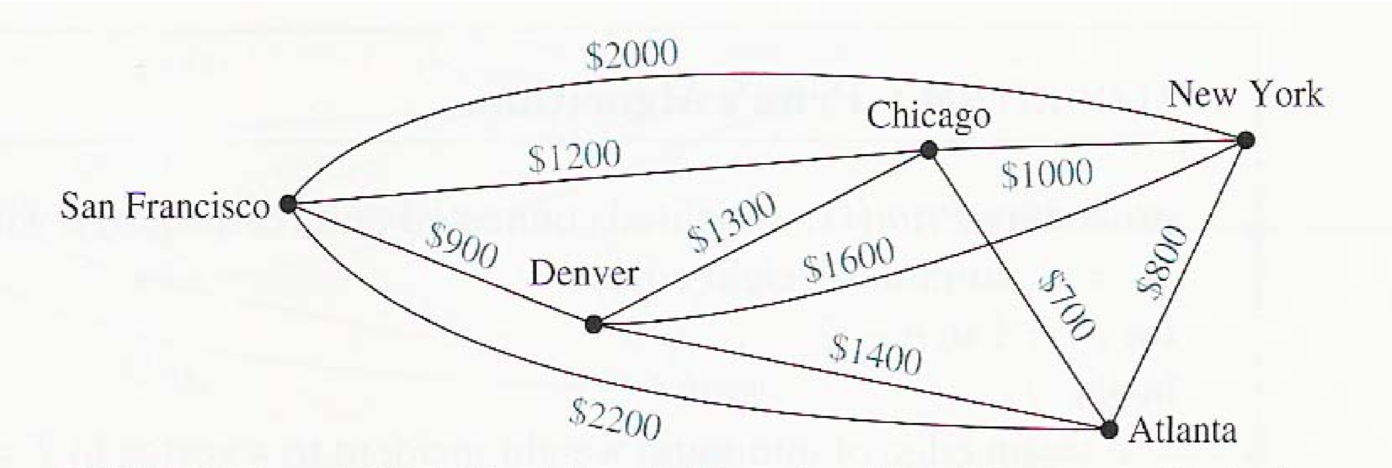

FIGURE 1 **A Weighted Graph Showing Monthly Lease Costs for Lines in a Computer Network.**

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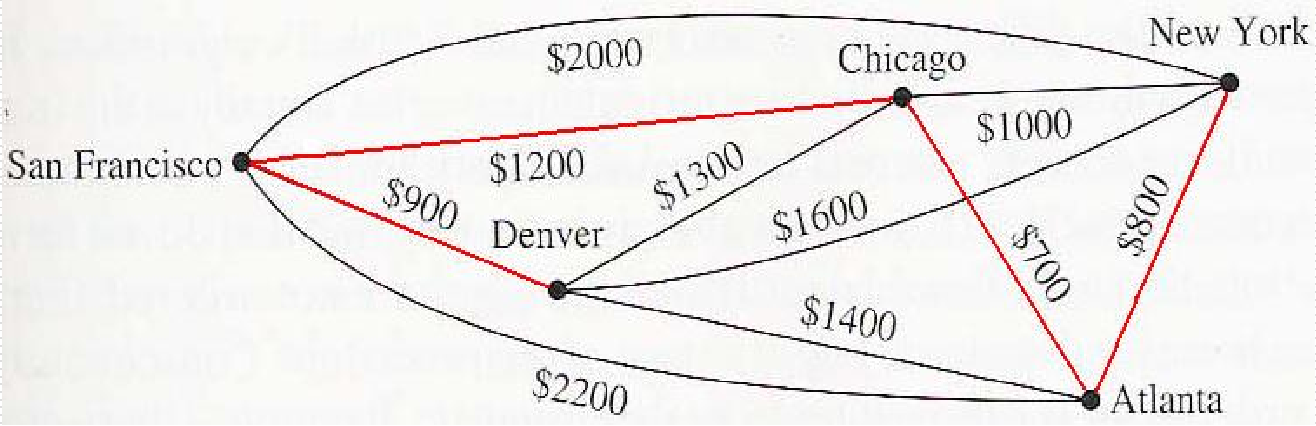
Example Using Prim's Algorithm

- Choosing and initial edge of minimum weight.
- Successively adding edges of minimum weight that are incident to a vertex in a tree and do not form a simple circuit.

Example Using Prim’s Algorithm



Choice	Edge	Cost
1	{Chicago, Atlanta}	$700
2	{Atlanta, New York}	$800
3	{Chicago, San Francisco}	$1200
4	San Francisco, Denver}	$900
	Total:	$3600

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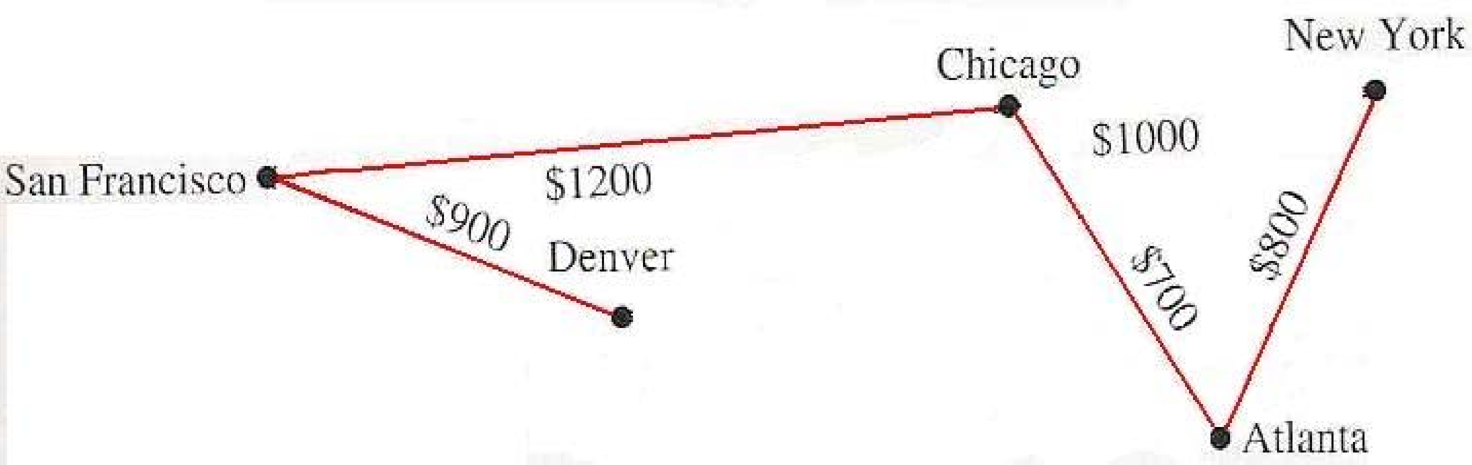
Example Using Prim's Algorithm



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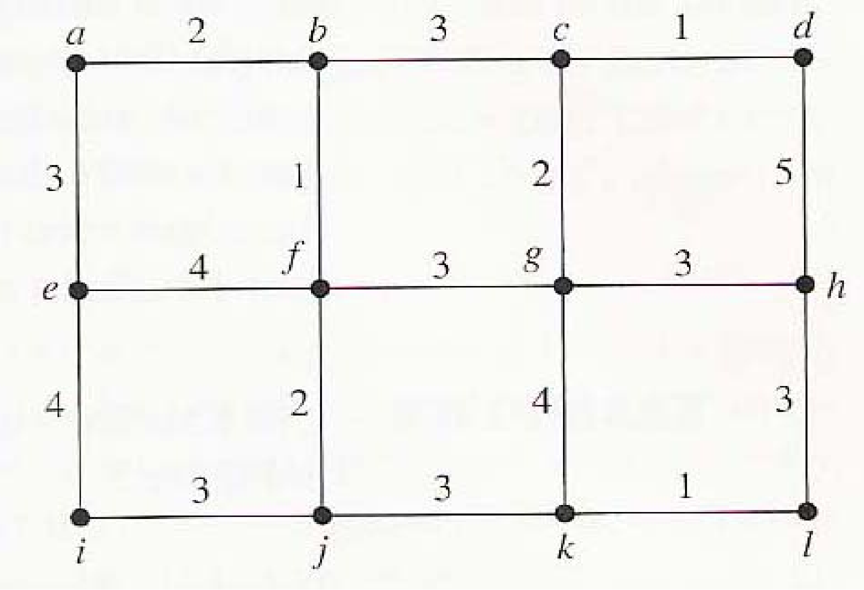
Example Using Prim’s Algorithm

- Use Prim’s algorithm to find a minimum spanning tree in the following graph.



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Example Using Prim's Algorithm

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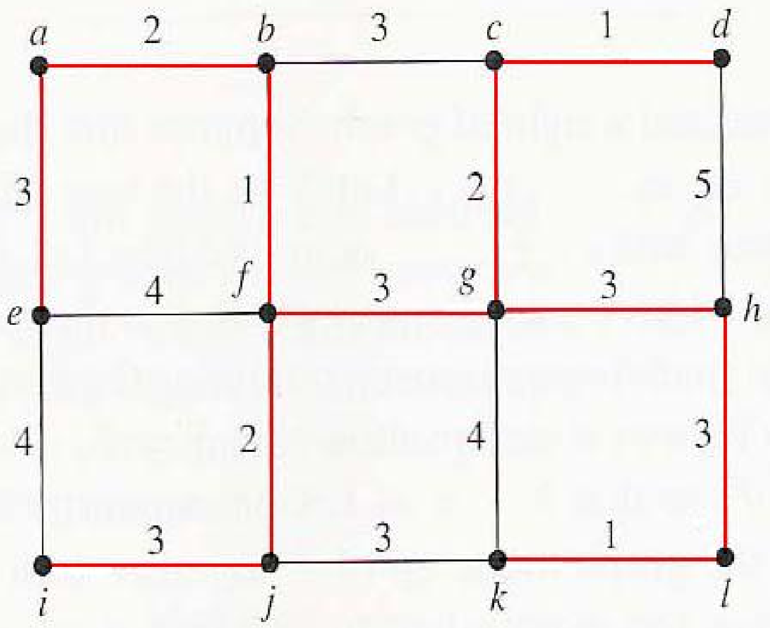

Choice	Edge	Weight
1	$\{b, f\}$	1
2	$\{a, b\}$	2
3	$\{f, j\}$	2
4	$\{a, e\}$	3
5	$\{i, j\}$	3
6	$\{f, g\}$	3
7	$\{c, g\}$	2
8	$\{c, d\}$	1
9	$\{g, h\}$	3
10	$\{h, l\}$	3
11	$\{k, l\}$	1
	Total:	24

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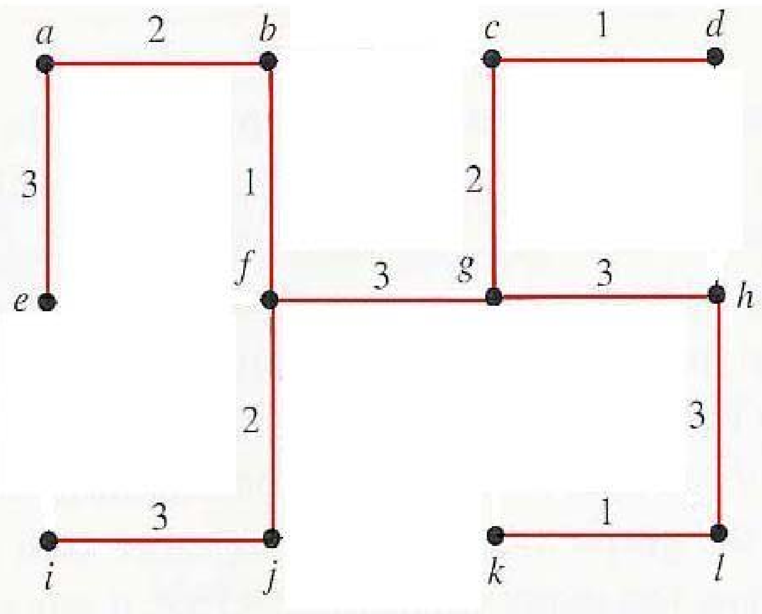
Example Using Prim's Algorithm

- Use Prim's algorithm to find a minimum spanning tree in the following graph.



Choice	Edge	Weight
1	{*b*, *f*}	1
2	{*a*, *b*}	2
3	{*f*, *j*}	2
4	{*a*, *e*}	3
5	{*i*, *j*}	3
6	{*f*, *g*}	3
7	{*c*, *g*}	2
8	{*c*, *d*}	1
9	{*g*, *h*}	3
10	{*h*, *l*}	3
11	{*k*, *l*}	1
	Total:	24

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Kruskal’s Algorithm

- This algorithm finds a minimum spanning tree in a connected weighted graph with n > 1 vertices.

Kruskal’s Algorithm

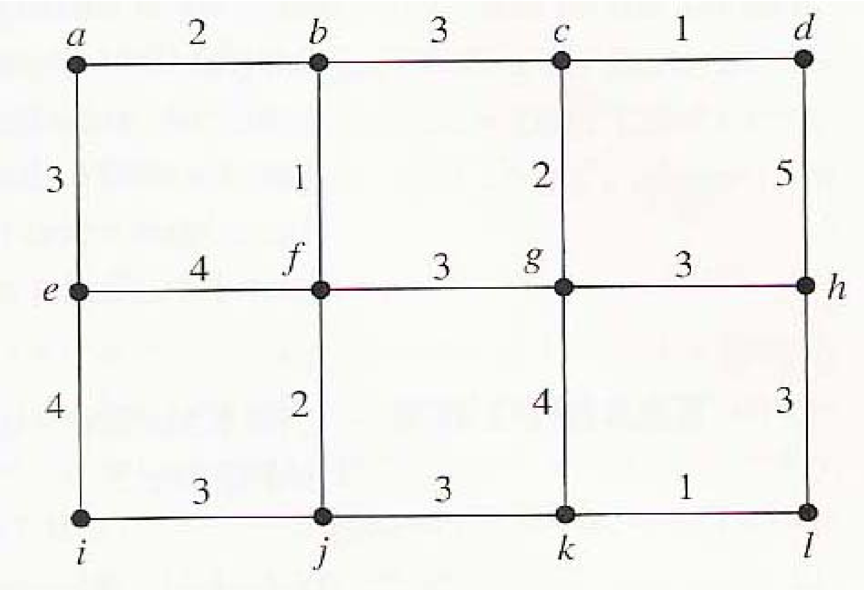
```
Step 1: Find an edge of least weight and call this $e_1$.
    Set $k = 1$.
Step 2: While $k < n$

        if there exits an edge e such that
        $\{e\} \cup \{e_1,. e_2, \ldots, e_k\}$ does not contain a circuit

          * let $e_{k+1}$ be such an edge of least weight;
            replace k by $k + 1$;
    else output $e_1, e_2, \ldots, e_k$ and stop
    end while
```

Example Using Kruskal’ Algorithm

- ☐ Use Kruskal’s algorithm to find a minimum spanning tree in the following weighted graph.



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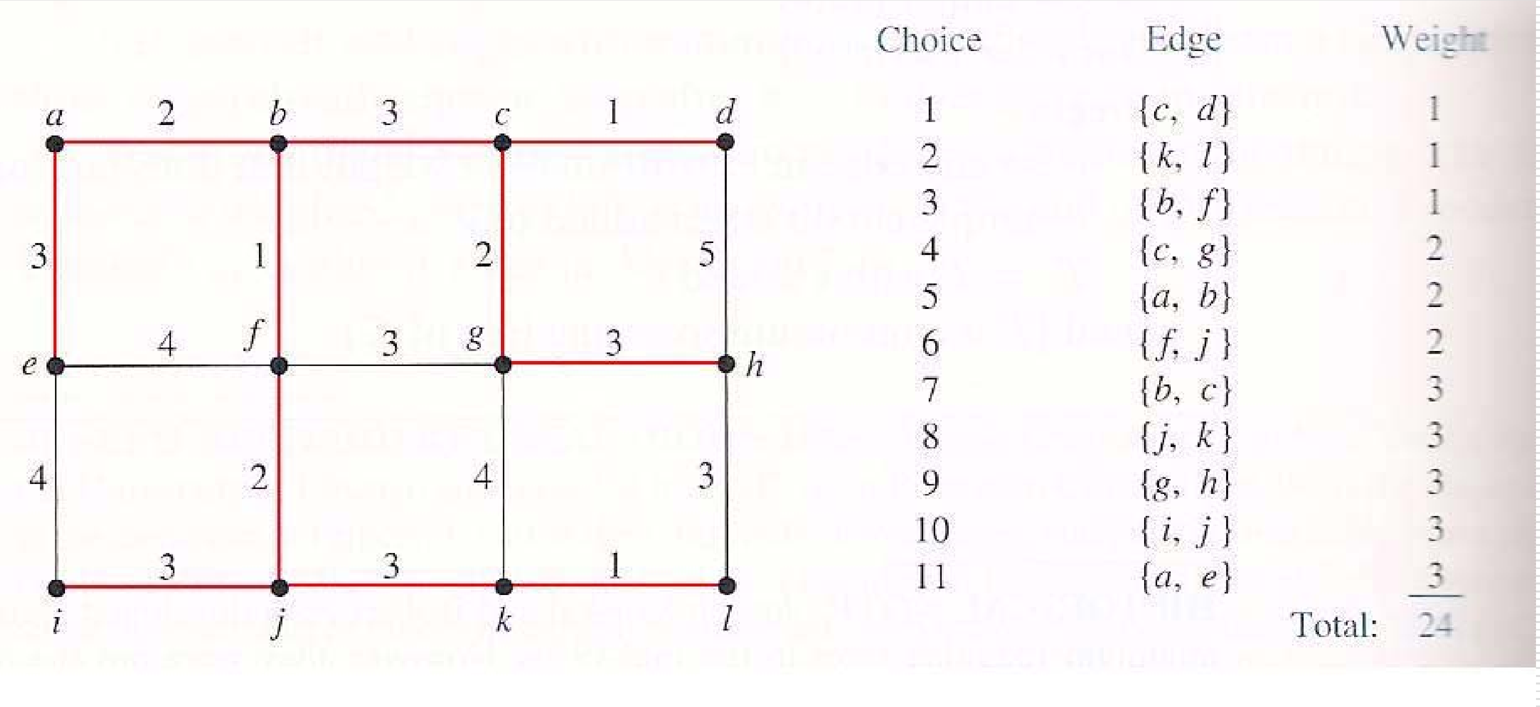
Example Using Kruskal’ Algorithm

- Use Kruskal’s algorithm to find a minimum spanning tree in the following weighted graph.



Choice	Edge	Weight
1	{*c*, *d*}	1
2	{*k*, *l*}	1
3	{*b*, *f*}	1
4	{*c*, *g*}	2
5	{*a*, *b*}	2
6	{*f*, *j*}	2
7	{*b*, *c*}	3
8	{*j*, *k*}	3
9	{*g*, *h*}	3
10	{*i*, *j*}	3
11	{*a*, *e*}	3
	Total:	24

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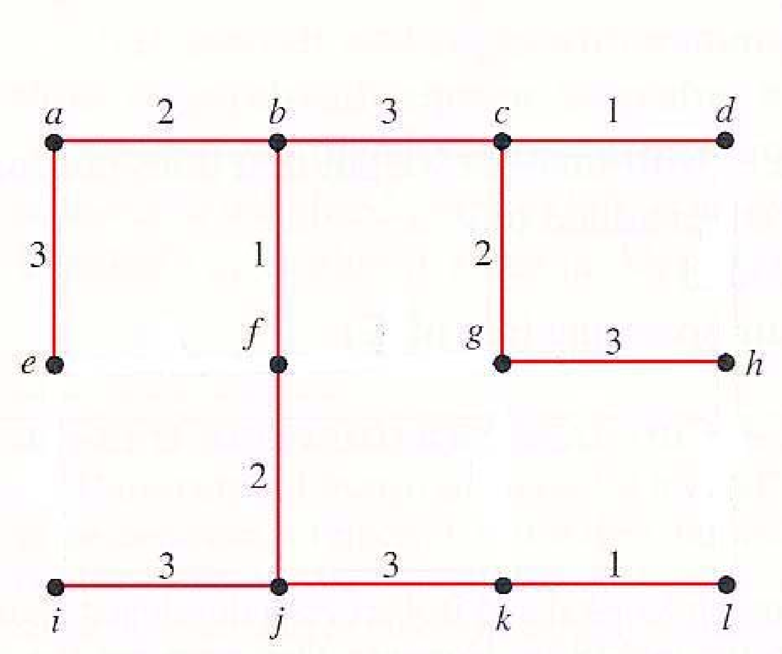
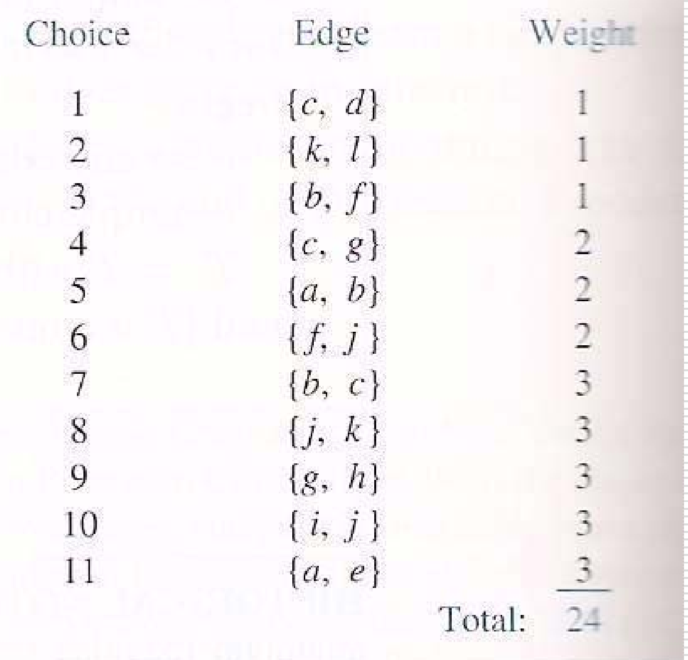
Example Using Kruskal' Algorithm

- Use Kruskal's algorithm to find a minimum spanning tree in the following weighted graph.





Choice	Edge	Weight
1	{*c*, *d*}	1
2	{*k*, *l*}	1
3	{*b*, *f*}	1
4	{*c*, *g*}	2
5	{*a*, *b*}	2
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7	{*b*, *c*}	3
8	{*j*, *k*}	3
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10	{*i*, *j*}	3
11	{*a*, *e*}	3
	Total:	24

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Digraphs

- A digraph is a pair (V, E) of sets, V nonempty and each element of E an ordered pair of distinct elements of V.
- The elements of V are called **vertices** and the elements of E are called **arcs**.

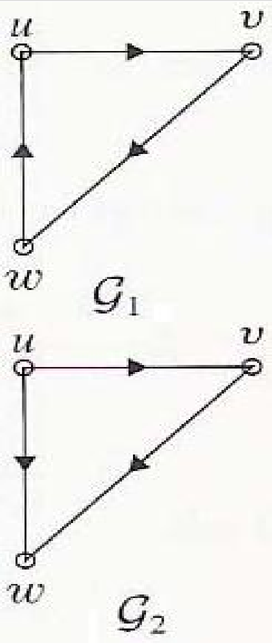
Digraphs

- The same terms can be used for graphs and digraphs.
- The exception: In a digraph we use term arc instead of edge.
- An arc is an ordered pair (u, v) or (v, u).
- An edge is an unordered pair of vertices {u, v}.

Digraphs

- The vertices of a graph have degrees, a vertex of a digraph has an indegree and outdigree.
- Indegree is the number of arcs directed into a vertex.
- Outdegree is the number of arcs directed away from the vertex.

Examples: Digraphs

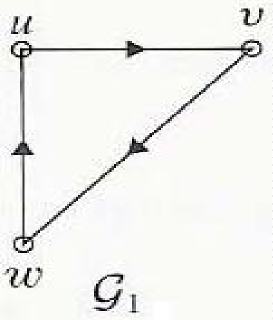


- G_1: u has outdegree 1,
  v has outdegree 1,
  w has outdegree 1
- G_2: u has outdegree 2,
  v has outdegree 1,
  w has outdegree 0

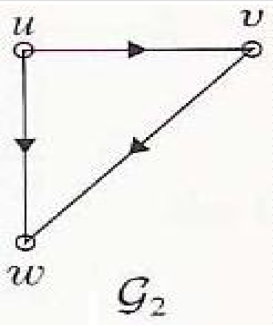
- G_1 and G_2 are not isomorphic.

“Discrete Mathematics with Graph Theory.” Fifth Edition, by E. G. Goodaire ane M. Parmenter Prentce Hall, 2006. pag 344

Examples: Digraphs



- G_1 is Eulerian because uvwu is an Eulerian circuit and a Hamiltonian cycle.



- G_2 has neither an Eulerian nor a Hamiltonian cycle, but it has a Hamiltonian path uvw.

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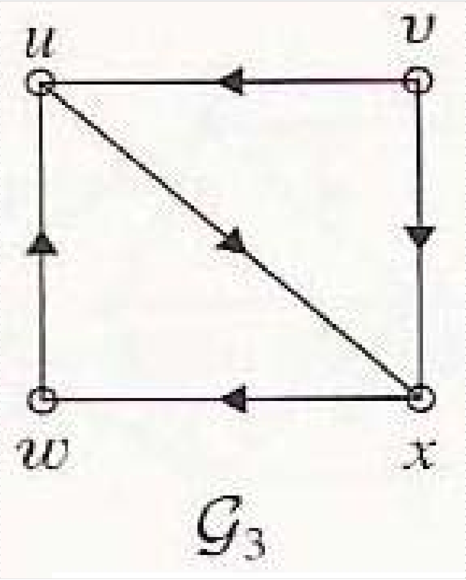
Examples: Digraphs



- G_3 has vertices u and x with indegree 2 and outdegree 1.
- Vertex v has indegree 0 and outdegree 2 and vertex w has indegree 1 and outdegree 1.

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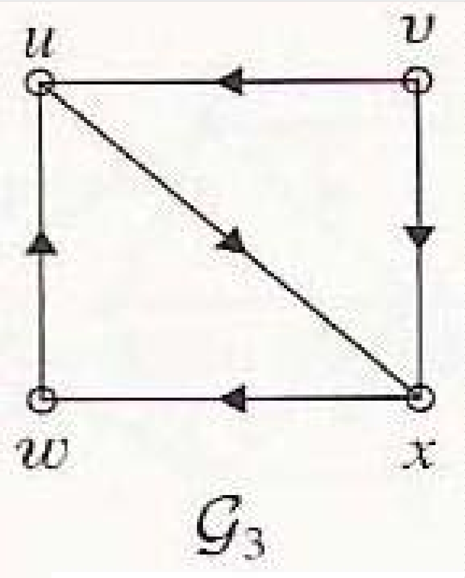
Examples: Digraphs



- The indegree sequence is 2, 2, 1, 0, and the outdegree sequence is 2, 1, 1, 1.
- The sum of the indegrees of the vertices equals the sum of the outdegrees of the vertices is the number of arcs.

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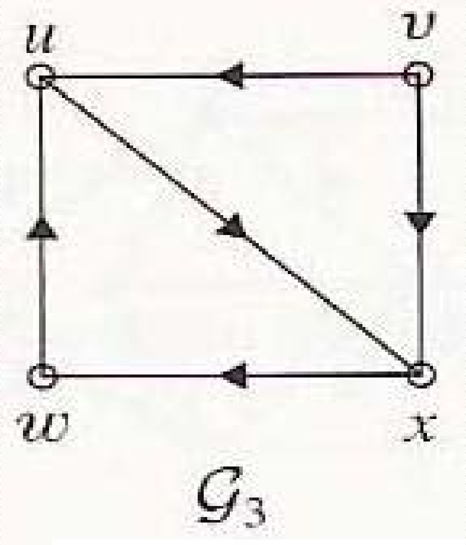
Examples: Digraphs



- G_3 is not Hamiltonian because vertex v has indegree 0.
- There is no way of reaching v on a walk respecting orientation edges, no Hamiltonian cycle can exist.

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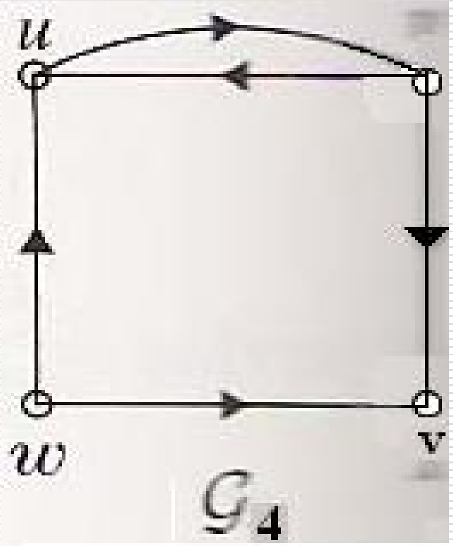
Examples: Digraphs



- ☐ G_4 is not Hamiltonian because vertex x has outdegree 0, so no walk respecting orientations can leave x.

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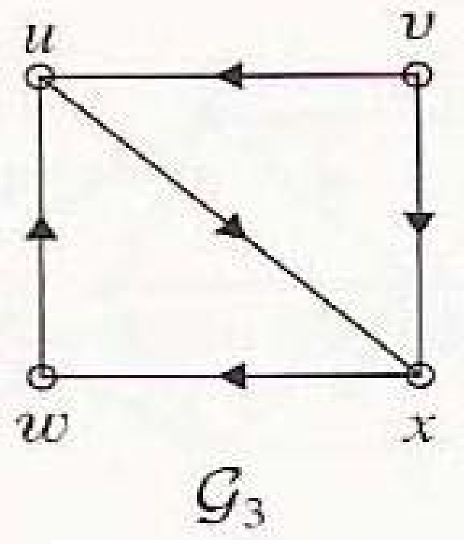
Digraphs

- A digraph is called strongly connected if and only if there is a walk from any vertex to any other vertex that respects the orientation of each arc.
- A digraph is Eulerian if and only if it is strongly connected and, for every vertex, the indegree equals the outdegree.

Examples: Digraphs



- G_3 is not Eulerian. It is not strongly connected (there is no way to reach v).
- The indegrees and outdegrees of three vertices (u, v, and x) are not the same.

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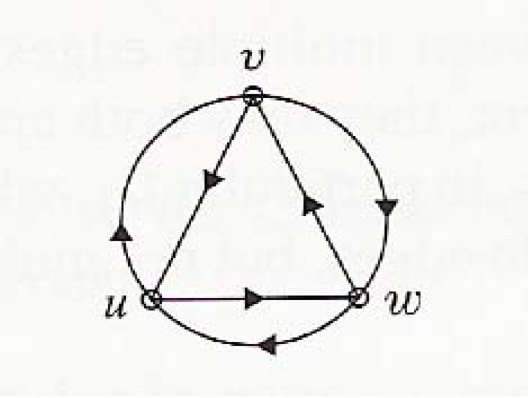
Examples: Digraphs

- This digraph is Eulerian.



- It is strongly connected (there is a circuit uvwu that permits travel in the direction of arrows between two vertices

- The indegree and outdegree of every vertex are 2 (an Euler circuit uwvuvwu).

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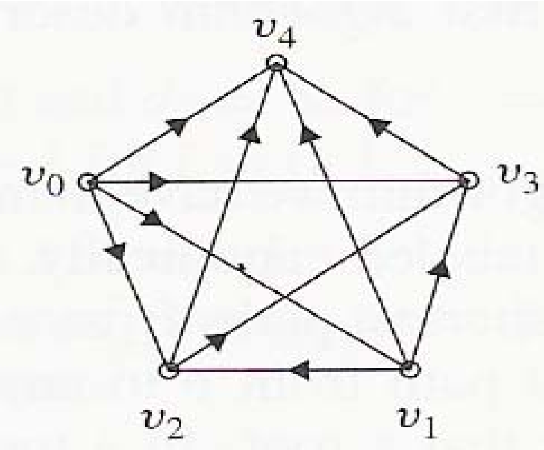
Acyclic Digraphs

- A directed graph is a acyclic if it contains no directed cycles.



- This digraph is acyclic.
- There are no cycles.
- There is never an arc on which to return to the first vertex.

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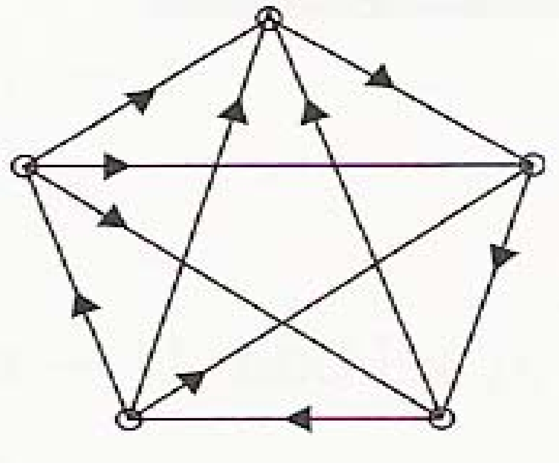
Acyclic Digraphs

- A directed graph is a acyclic if it contains no directed cycles.



- This digraph is not acyclic.
- There are cycles.

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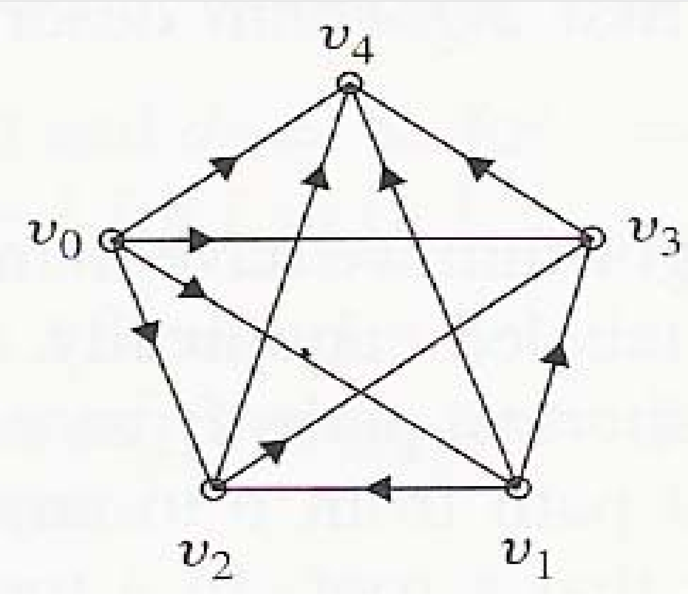
A Canonical Ordering

- A labeling v_0, v_1, ..., v_{n-1} of the vertices of a digraph is called canonical if the only arcs have the form v_iv_j with $i < j$.
- A canonical labeling of vertices is also called a canonical ordering.
- A digraph has a canonical ordering of vertices if and only if it is acyclic.

A Canonical Ordering



- A digraph has a canonical ordering of vertices if and only if it is acyclic.
- A digraph is acyclic if and only if it has a canonical labeling of vertices

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Strongly connected Orientation

- To orient or to assign an orientation to an edge in a graph is to assign a direction to that edge.

- To orient or assign an orientation to a graph is to orient every edge in the graph.

- A graph has a strongly connected orientation if it is possible to orient it in such a way that the resulting digraph is strongly connected.

Depth-First Search

- Depth-first search is a simple and efficient procedure used as the for a number of important computer algorithms in graphs.
- We can build a spanning trees for a connected simple graph using a depth-first search.

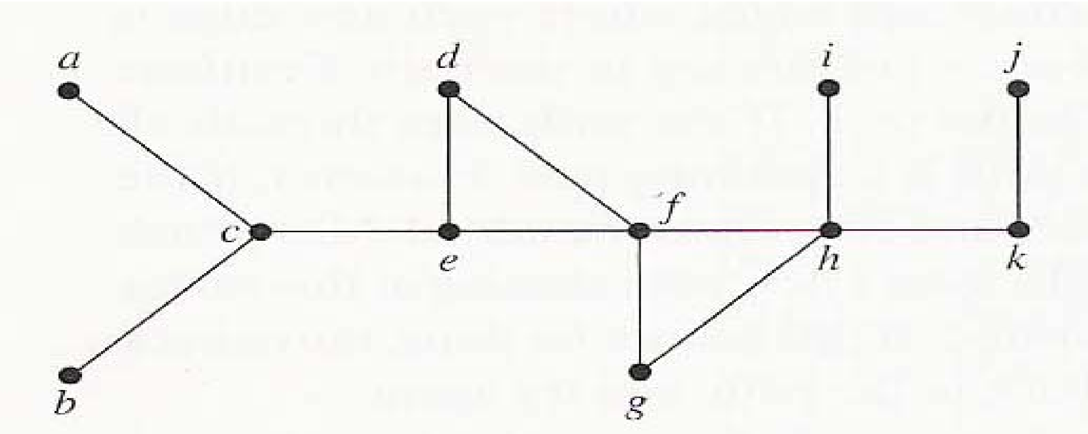
Depth-First Search Algorithm

- Let G be a graph with n vertices.

```
Step 1. Choose any vertex and label it 1. Set k = 1.
Step 2. While there are unlabeled vertices
            if there exists an unlabeld vertex adjacent to k,
                assign to it the smallest unused label l from
                the set {1, 2, ..., n} and set k = l
            else if k = 1 stop;
                else backtrack to the vertex l from which k
            was labeled and set k = l.
Step 3. end while.
```

Example: Depth-First Search

- Use a depth-first search to find a spanning tree for the graph G.



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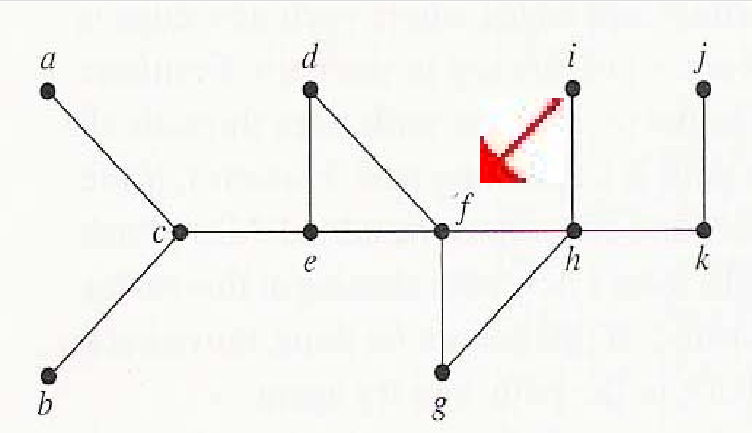
Example: Depth-First Search



- Arbitrary start with vertex f.
- A path is built by successively adding edges incident with vertices not already in the path, as long as possible

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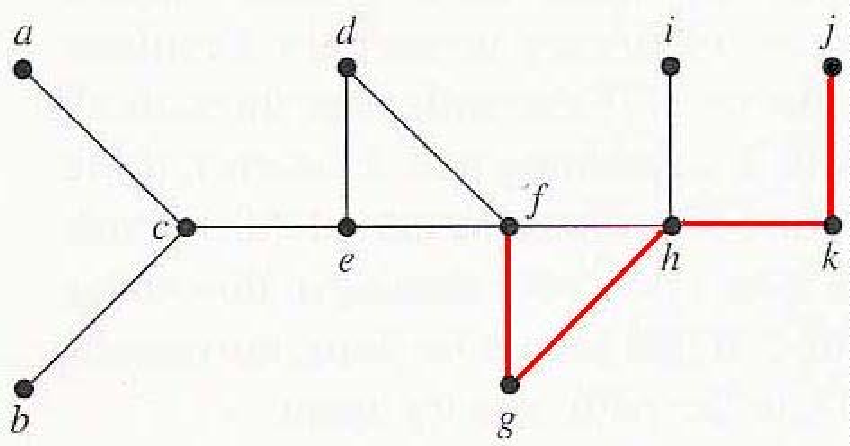
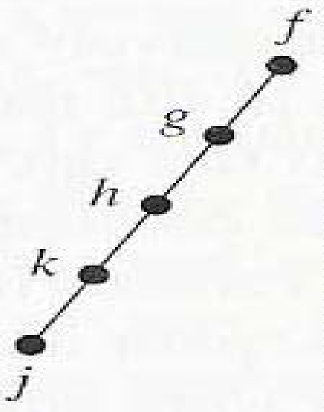
Example: Depth-First Search





- From f create a path f, g, h, j
- (other path could have been built).

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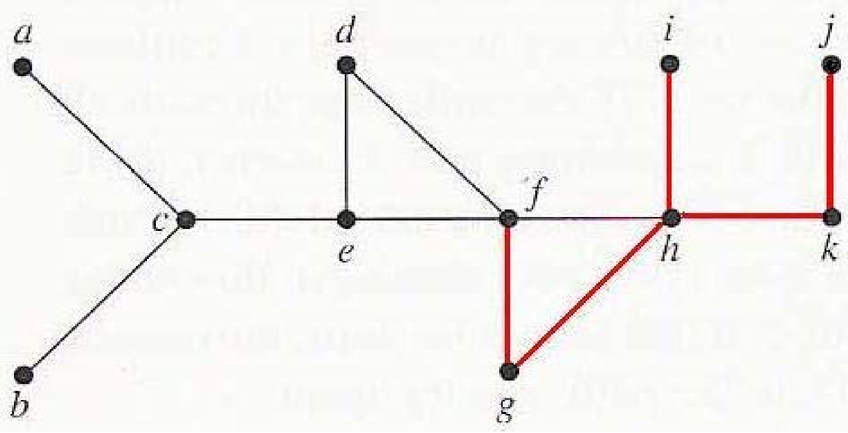
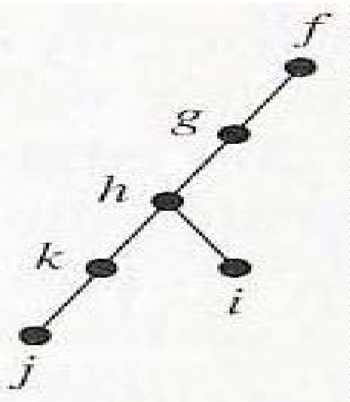
Example: Depth-First Search





- Next, backtrack to k. There is no path beginning at k containing vertices not already visited.
- So, backtrack to h.
- Form the path h, i.

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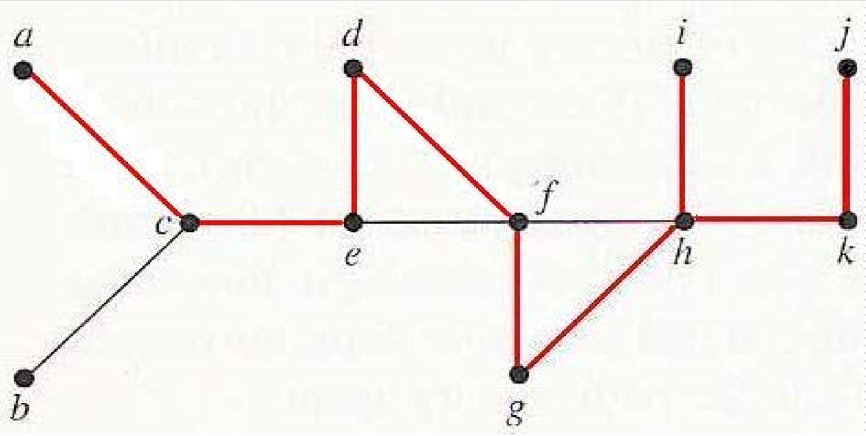
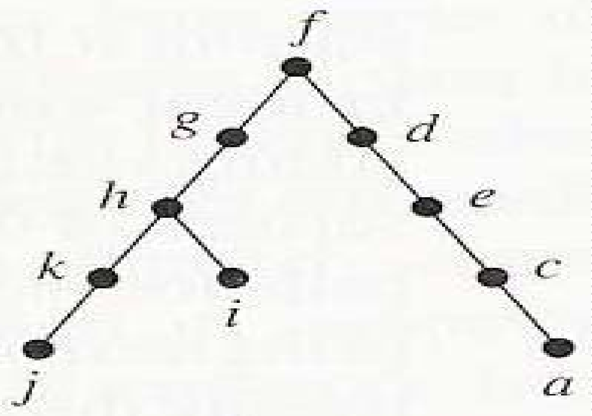
Example: Depth-First Search





- Then, backtrack to h, and then to f.
- From f build the path f, d, e, c, a.

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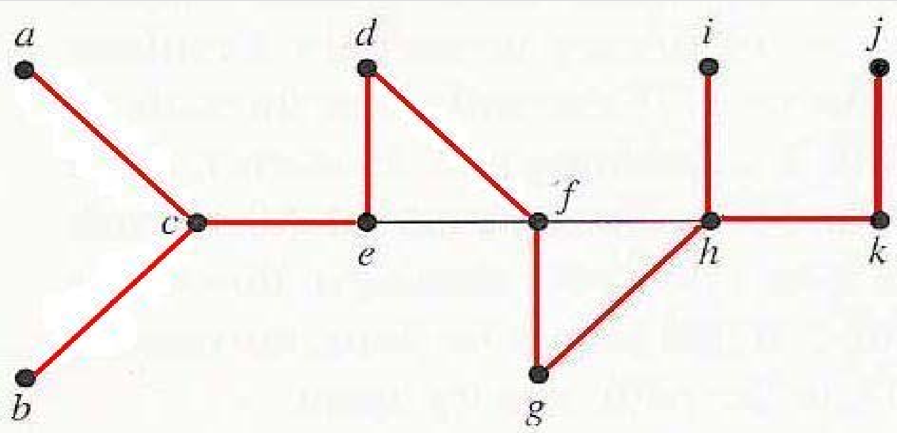
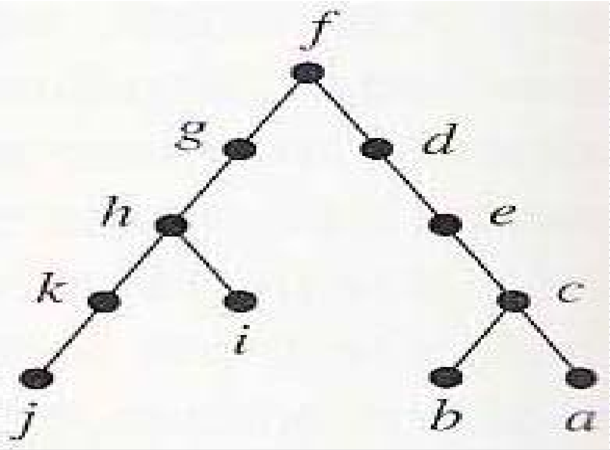
Example: Depth-First Search





- Then, backtrack to c, and form the path c, b.
- The result is the spanning graph.

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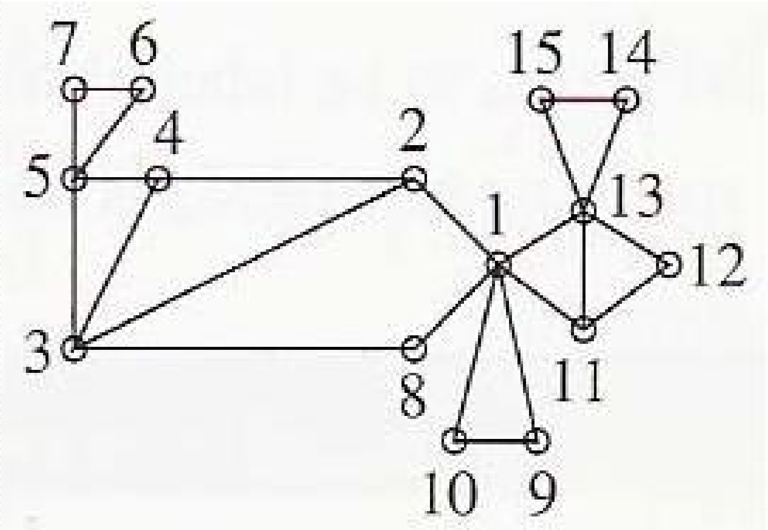
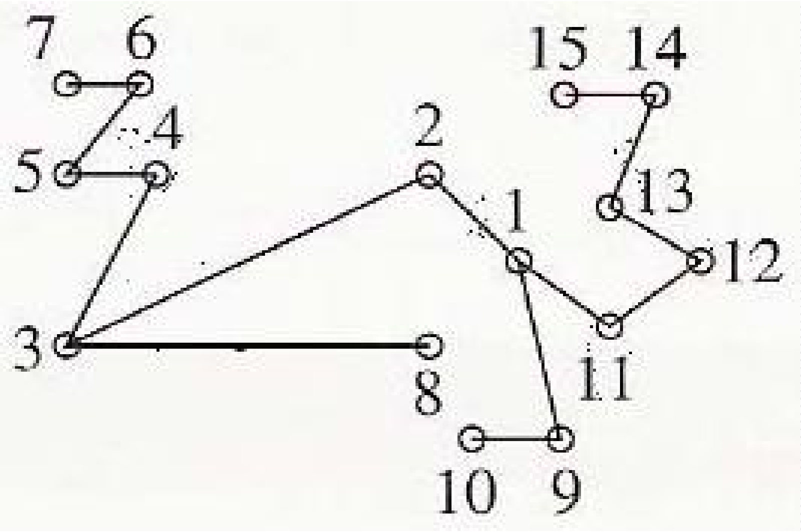
Example: Depth-First Search

- Use a depth-first search to find a spanning tree for the graph G.





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Topics covered

- Trees and their properties.
- Spanning trees and minimum spanning trees algorithms.
- Depth-First Search.

Reference

- “Discrete Mathematics with Graph Theory”, Third Edition, E. Goodaire and Michael Parmenter, Pearson Prentice Hall, 2006. pp 370-410.

Reference

- “Discrete Mathematics and Its Applications”, Fifth Edition, Kenneth H. Rosen, McGraw-Hill, 2003. pp 631-694.